THE MATHEMATICS STUDENT

A QUARTERLY DEDICATED TO THE SERVICE OF
STUDENTS AND TEACHERS OF MATHEMATICS IN INDIA

Edited by
A. NARASINGA RAO, M.A., L.T.

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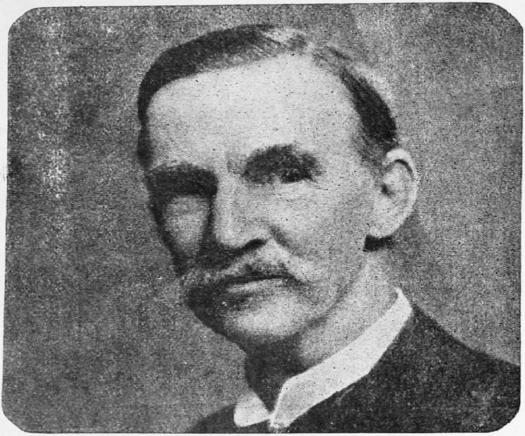

ERNEST WILLIAM HOBSON.

Born 1856. Died 1933.

(From the Illustrated London News.)

THE MATHEMATICS STUDENT

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FOREWORD.

THE Indian Mathematical Society has completed the first twenty-five years of its existence and, at the invitation of the Bombay University, its Eighth Conference and the Silver Jubilee Celebrations took place at Bombay in December 1932.

The Society feels that the time is come for it to enlarge its activities so as to serve the needs of students and teachers of Mathematics, especially of the collegiate grade and of young research workers, even more fully than has been possible hitherto. With this aim in view, it has been decided to issue two quarterlies in place of the JOURNAL, roughly corresponding to Part I and Part II. The first of these will commence publication from 1934, and in its place will be issued in 1933 the Jubilee Memorial Volume containing an account of the Bombay Conference and Jubilee Celebrations and the full texts of the papers communicated thereto. The second quarterly is now issued under the title THE MATHEMATICS STUDENT.

Every attempt will be made in it to give a full and varied programme of matter. It will seek to stimulate intetest, encourage wide reading and a critical appreciation of results. It will be the official organ for all announcements by the Society and will attempt to keep its readers in touch with the mathematical world of India and abroad. Finally, it will devote itself, wholeheartedly, to raise the standard of Mathematical instruction in the country by providing a medium for the exchange of ideas, and for drawing attention to defects in syllabuses and methods of instructions and suggestions for their improvement.

The success of the new venture will be largely dependent on the sustained and enthusiastic collaboration and good-will of all lovers of Mathematics in India and our friends abroad.

May I appeal to all fellow-students of mathematics to help me with their valuable contributions, suggestions and criticisms, so that this periodical may mark a further step towards the "advancement of mathematical study and research in India,"—the goal of the Society.

A. NARASINGA RAO.

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SOME GLIMPSES OF ANCIENT HINDU MATHEMATICS.

By A. A. KRISHNASWAMY AYYANGAR.

In ancient India, Mathematics was first developed as a Vedanga or a limb of the Vedas, an art subserving a religious purpose such as the construction of sacrificial altars. With rare intuitive ability the early Hindus grasped quickly the fundamentals of the subject and embodied them in short verses. To them, the world owes some of the basic ideas in mathematics—the place-value system of notation in Arithmetic, the generalizations of Algebra, the sine-function in Trigonometry, and the foundations of Indeterminate Analysis. India may claim to be the birth-place of Algebra and Arithmetic.

To a casual student of ancient Hindu Mathematics, the presentation may appear fragmentary and incoherent. The early texts contain merely rules, results and, sometimes, a number of problems but rarely a fully worked out mathematical argument. The Indian tradition was to make the text as brief as possible, so that the whole might be easily learned by heart thus enabling the ancient scholars to escape from the slavery of books.

The mathematical works of the early Hindus have a remarkable parallel in Ramanujan's note-books and methods of work. Dr. G. N. Watson remarks* that his (Ramanujan's) procedure in his note-books is in accordance with the traditional custom, for the early Hindus were not in the habit of preserving their proofs, though they doubtless reasoned out most or all of their discoveries. The following words of Dr. Littlewood,† about Ramanujan apply equally to the ancient Hindu mathematicians: 'Ramanujan's intuition worked in analogies sometimes very remote, and to an astonishing extent by empirical induction from particular numerical cases. He was not interested in rigour which, for that matter, is of secondary importance in analysis and can be supplied, given the real idea, by any competent professional. The

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* Lecture delivered on 5-2-1931 on 'Ramanujan's Note-books': reprinted in the *Journal of the London Mathematical Society*, Vol. VI. p. 2

† Review of the *Collected Papers of Srinivasa Ramanujan* by Littlewood in *Nature*, dated 27-4-1929.

clear-cut idea of what is meant by a proof, he perhaps does not possess at all. If a significant piece of reasoning occurred somewhere and the total mixture of evidence and intuition gave him certainty, he looked no further.'

The earliest Mathematical writings.

Let us start by taking a glimpse of the Vedanga Jyotisha, avowedly the oldest Indian work bearing on astronomy. The text is very corrupt and obscure in spite of Weber's* efforts at elucidation. The style is characterised by an enigmatical brevity, strange archaisms and total want of connection between consecutive verses. It expounds the doctrine of a cycle or Yuga of five years beginning with the winter solstice and the new-moon of Magha. According to this text, a year contains 366 days, and a Yuga contains 61 Sayana months, 66 lunar months, or 67 nakshatra months. Rules are given for calculating the relative lengths of day and night for any nycthemeron. The lengths of the longest and shortest days, according to the Jyotisha, are 18 and 12 muhurtas respectively, a nycthemeron comprising 30 muhurtas. The Jyotisha mentions the positions of the moon at different parts of the year and the time of day when a parvan (or a semi-lunar month) is finished. It is not known what additional information the unintelligible portion of the work may contain. Anyhow it is evident from the intelligible calculations given therein that the early Hindus were well-acquainted with arithmetical manipulations, including fractions.

Hindu Contributions to Arithmetic.

The next earliest mathematical document of the Hindus is probably the Sulva-sutras (not later than 200 A.D.) or the rules of measurement by cords, which deal with the construction of sacrificial altars. This work naturally reminds one of the Harpedonaptæ or the rope stretchers of Egypt. As it has a purely religious purpose in view, it gives bare directions for the construction of squares, rectangles, triangles and parallelograms with given specifications. Among other things it contains :—

(1) The first arithmetical solution of the indeterminate equation $x^2 + y^2 = z^2$ in the form

$$(mn)^2 + \left(\frac{m^2 - n^2}{2}\right)^2 = \left(\frac{m^2 + n^2}{2}\right)^2$$

* 'Uber den Vedakalendar, namens Jyotisham' by A. Weber in the *Transactions of the Royal Academy of Sciences*, Berlin, 1862.

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and (2) the rational approximations corresponding to

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$$

$$\frac{\sqrt{\pi}}{2} = 1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}.$$

According to G. R. Kaye,* this implies a knowledge of the process of converting a fraction into partial fractions with unit numerators, a knowledge most certainly not possessed by the composers of the Sulva-sutras. We shall see presently how the premise is wrong and the conclusion is irrelevant.

In the year 1881 A.D., a work on Arithmetic was discovered by a peasant near a village called Bakshali in the North-West Frontier of India. It contains a second approximation to the square-root in the form

$$\sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{(r/2a)^2}{2(a + r/2a)}.\dagger$$

Incidentally we may note that this rule also finds a place in an Arab work of the 12th century.‡ The result is easily deduced by applying, twice, the usual rule for the first approximation or the usual square-root rule found in all the Indian Arithmetical works at least from Aryabhata onwards, a rule which is independent of the kind of notation used, positional or non-positional. Brahmagupta gives an application of the square-root rule to a non-positional case.§ It is not improbable therefore that the Sulva-sutra approximations were also obtained by using the

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* *Indian Mathematics*, by G. R. Kaye, p. 7.

† The Bakshali manuscript, by G. R. Kaye; *Archæological Survey of India*, New Imperial Series, Vol. XLII, Pts. I and II.

‡ P. 254 foot-note in Smith's *History of Mathematics*, Vol. II.

§ Verses 64, 65, p. 213, *Brahmasiddhanta* of Brahmagupta, edited by Sudhakara Dvivedi, Benares, 1902.

square-root rule thus :

$$
\begin{array}{r|l|l}
 & 2 & 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \\
 & 1 & \\
\hline
 & 1 & \\
2 + \frac{1}{3} & 7/9 & \\
\hline
2 + \frac{2}{3} + \frac{1}{3.4} & 2/9 & \\
 & 2/9 + \frac{1}{9.16} & \\
\hline
 & -\frac{1}{9.16} & \\
2 + \frac{2}{3} + \frac{2}{3.4} - \frac{1}{3.4.34} & & \\
 & -\frac{1}{9.16} + \frac{1}{3^2.4^2.34^2} & \\
\hline
 & \ldots\ldots\ldots &
\end{array}
$$

As Arithmetic was the *forté* of the Hindus, all geometrical problems were first analysed as problems in Arithmetic and then expressed in geometrical language. Many geometrical theorems such as Brahmagupta's well-known formula for the area of a cyclic quadrilateral were apparently discovered arithmetically. Indeed what is now known as the Pythagorean theorem was recognised by the early Hindus as a property* of two square numbers (say a^2, b^2) being together equal to a third square (c^2) and when cords of those lengths (a, b, c) were put together (accidentally or otherwise) to form a quadrilateral with a diagonal equal to c and adjacent sides a, b, a rectangle was the result revealing thus the existence of a right-angle. Thus a sort of arithmetical intuition was at the back of a geometrical discovery. The phraseology of the Sulva-sutras betrays this :

The diagonal of a rectangle produces a square equal to the sum of the squares produced by the longer and the shorter sides respectively.

The proof of this theorem, as noted by Bhaskara (1150 A.D.) is based on the property of similar triangles assumed as an axiom.†

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* Several such sets of squares are given in the Sulva-sutras ;

$3^2 + 4^2 = 5^2$; $5^2 + 12^2 = 13^2$

$7^2 + 24^2 = 25^2$; $8^2 + 15^2 = 17^2$

$12^2 + 16^2 = 20^2$; $12^2 + 35^2 = 37^2$

† For an exposition of Bhaskara's proof, *vide* p. 499. '*New Geometry*' by the present writer.

The age of Aryabhata.

Taking a leap over a few centuries from the period of the Sulvasutras, we come to the age of Aryabhata, Varahamihira, Brahmagupta, and the unknown author of the *Suryasiddhanta* (400-650 A.D.)

According to Thibaut, Aryabhata is unaccountably placed by tradition at the head of the Indian mathematicians. To my mind he appears to be the last of the earlier school. His style is reminiscent of the Sutra period. It is condensed to the point of unintelligibility, and coldly business-like. But the importance of the work lies in the fact that it appears to be the first to give a form and an individuality to the scattered bits of mathematical knowledge that existed before his time. The role of the ***Aryabhatiyam*** in giving a definite bias to Indian mathematics has its historical parallel in two other great ancient mathematical compositions—the ***Elements*** of Euclid and the ***Arithmetica*** of Diophantus

Aryabhata's work attracted much attention even outside India. His treatise on Algebra was translated into Latin by G. de Lunis, an Italian mathematician of the 13th century, and there is a manuscript copy of the translation in the Bibliotheque Nationale de Florence.* In 1874, Dr. Kern brought out the first edition of the text of the ***Aryabhatiyam*** with the long commentary of Paramadiswara. In 1879 Rodet gave a French translation of the ***Ganita-pada***, the mathematical portion of the text, with very valuable and interesting notes. Thibaut, in 1899, gave a summary of the literature about Aryabhata. Another translation of Aryabhata's work with critical notes was published in 1930, by Walter Eugene Clark of Harvard University.† All this attention was well-deserved as he was a pioneer in several respects.

He was an innovator in astronomy and paid dearly for it at the hands of Brahmagupta, who heaped unreasonable abuses on him. He was also the first in India, to hint the daily rotation of the earth on its axis

In his work, again, we find for the first and the last time a new alphabetic notation adopting both the consonants and vowels to represent

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* A reference to this is given in *L'Intermediare-des-mathematicians*, July and August, 1919.

† Vide *The Aryabhatiyam of Aryabhata* by Walter Eugene Clark-(1930) The Unversity of Chicago, Illinois.

even large numbers in a very concise way,

e.g., ङिशिबुण्लृख्षृ = 1582237500.

Probably this notation marks an important stage in the development of the numeral notation and is a precursor of the Indian decimal notation. It suggests a method of using the same symbol, say क, with such variations as कि, कु, etc., to denote a power of a hundred multiplying the intrinsic value of the consonant. In other words, the vowels are suggestive of the local value and the consonants of the intrinsic value in this scheme of notation. To proceed from this stage to the place-value stage it required only to drop the vowels and make the position itself indicate which power of ten was intended. This would require setting apart a symbol for zero. But it took many long centuries to recognise that the zero was also a numeral on a par with other numerals, and that a separate symbol was necessary to denote it. It was only in the eleventh century, after the decimal notation with its place-value and zero had become definitely established that another genius arose to re-adopt the alphabet to the new notation. In this connection it must be observed that the alphabetic notation in India was felt more or less as a necessity owing to the exigencies of condensed metrical composition and therefore there is a greater likelihood of its being indigenous to India than a loan from Greece or elsewhere.

Aryabhata's notation was never popular in India. Even his admirer Lalla abandoned it in favour of the more popular and the more ancient word-numerals. These could more easily be remembered and could secure better metrical euphony. The first glimpses of them are found in the Vedanga-Jyotisha. The anonymous author of *Surya-siddhanta* (not later than 5th century A.D.), and Varahamihira freely use these numeral words. But Brahmagupta of the 6th century A.D. was the first to explicitly enunciate this system in the following words:

"If you want to write one, express it by everything which is unique as the earth, the moon; two by everything which is double, as for example black and white; three by everything which is threefold; the nought by heaven, the twelve by the names of the Sun."

Does not this bring us very near the modern definition of number, for, according to Bertrand Russell, 'a number is any collection which is the number of one of its members, or more simply still, a number is anything which is the number of some class'?

Side by side with the popular word-numeration there was the equally popular Brahmi notation, used generally in inscriptions, and the two combined together to produce the modern system of numerals with the place-value and zero.*

The formulation of puzzle problems and their solution proved to be a new incentive to mathematical research in Ancient India. Regarding his miscellaneous problems Brahmagupta remarks: 'These questions are stated merely for pleasure. The proficient may devise a thousand others or may solve the problems of others by the rules given here. As the stars are obscured by the sun, so does the expert eclipse the glory of other astronomers in an assembly of people by proposing algebraic problems and still more by their solution.' The later mathematical works contain many puzzle problems mainly based on indeterminate analysis. To quote one instance, Bhaskara cites the following as due to earlier authors:

"Three traders with 6, 8, and 100 panas as capital respectively having bought fruits at a uniform rate, sold a part in lots and disposed of the remainder at one for 5 panas and thus became equally rich. What were the purchase and the sale-prices ?"†

The rule for solving such problems is enunciated by Vishnugupta, Mahaviracharya and others thus:

"The required sale-price is any number greater than the greatest of the given capitals, while the purchase-price is less than the product of the two sale-prices by unity."

It is interesting to note that the puzzle problems of Widmont and Blasius discussed by Glaisher‡ are perhaps the European recensions of the above Indian problem.

We give below a list of the important mathematical topics on which there are contributions from the ancient and the medieval Hindus:

(1) Surds, square-root of binomial and polynomial quadratic surds, Involution and Evolution.

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* For a detailed exposition of the growth of the so-called Hindu-Arabic numerals, *vide* an article under that name by the present writer in the *Quarterly Journal of the Mythic Society*, Vol. XVIII, No. 4, Vol. XIX, Nos. 1 and 2.

† *Jour. Ind. Math. Soc.*, Aug. 1914: 'A Classical Indian Puzzle Problem.'

‡ *Messenger of Mathematics*, Vol. LIII, No. 2.

(2) The Arithmetical and Geometrical Progressions and allied series.

(3) Simple and quadratic equations. (real roots) simultaneous equations in many variables. Equations reducible to quadratics.

Identities. Solutions of equations by means of identities.

Euler's Identity:

$$1 + a_1 + a_2 (1 + a_1) + \dots + a_n (1 + a_1) \dots (1 + a_{n-1}) = \prod_1^n (1 + a_r).$$

(4) Permutations and Combinations.

(5) Indeterminate equations of the first and the second degree.

(6) Numerical approximations for $\sqrt{n}$ and π. Approximations for a side of a regular polygon inscribed in a circle.

(7) Interpolation formulæ, especially in connection with the sine-table.

(8) Properties of triangles, quadrilaterals, circles,

(9) Mensuration of the sphere, the cone, and the pyramid.

(10) Rule of three terms (direct and inverse), rule of five, seven, nine and eleven terms; Chain-rule; Interest problems; Barter, Mixture; Motion; problems and relative velocity.

(11) Rule of Inverse operations (व्यस्तविधि).

(12) Values of π and the sine-table. Trigonometrical identities. Fundamental formulæ in spherical trigonometry.

(13) The principle of iteration (असकृत्कर्म) or successive approximations in solving some transcendental equations in astronomy. There is some analogy between this and Horner's method.

(14) The sun-dial and shadow problems. Eclipse problems based on a property of the circle.

(15) Magic squares.*

(16) Astronomy.

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* Methods of construction of magic squares appear to have been known in India even before the Christian era. A casual reference to this subject is given in Ganesa's gloss of Bhaskara's *Lilavati*, which is as late as 1520 A.D. Bhadraganita (mathematics of charms) or the construction of magic squares appears to have found a regular place in the Arithmetical treatises of Narayana and others. The Jhansi magic square is well-known (*Jour. Ind. Math. Soc.*, Feb., 1918.)

Hindu Contributions to Indeterminate Analysis.

Among the contributions of the Ancient Hindus to higher mathematics, their work on Indeterminate Analysis is the most outstanding.

Aryabhata was the first to indicate a method of getting integral solutions for the indeterminate equation of the first degree in the form,

$$mx + a = ny + b$$

(m, n, a, b being integers). Aryabhata proceeds by successive reductions* to simpler equations, until one is reached whose solution may be immediately guessed.† This is the true significance of the *mati* in Aryabhata's text and has been misunderstood by almost all the commentators, ancient and modern. Evidently they are confused between the logical and the psychological orders of evolution of mathematical ideas. Certainly the germ of the continued fraction was there in Aryabhata's method but was not clearly recognised by him It is a consolation to find that Bramagupta, who is no friend of Aryabhata, has thoroughly understood his adversary's rule and given a clearer exposition of the same. The later writers, Bhaskara and others, followed with improvements, alterations and extensions. This is, of course, to be expected. Viewed in this light, the evolution of linear indeterminate analysis from Aryabhata to Bhaskara and the later Aryabhata is quite natural.

Regarding indeterminate equations of the second degree of the type, $x^2 - Ny^2 = 1$ (N being a non-square integer), Brahmagupta

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* To illustrate: Take the equation $29x+15 = 45y+19$; *i.e.*, $x = y + \frac{16y+4}{29}$ $= y+y_1$ (say) so that $y = y_1 + \frac{13y_1 - 4}{16}$. This is the second indeterminate equation.

Next set $13y_1 - 4 = 16y_2$, so that $y_1 = y_2 + \frac{3y_2 + 4}{13}$. This is the third equation. At this stage, we notice that $y_2 = 3$ obviously is a solution, and '3' is Aryabhata's *mati*.

Hence, we proceed to find the values of y_1, y, and x in order by the "rule of inverse operations" set down in all Indian mathematical works and also in the *Aryabhatiyam* in particular. The successive steps of the reverse process are indicated by Aryabhata's rule, which, in the hands of later writers, is termed *Vallika Kuttakara*.

† *Vide* pp. 18—21, *Quarterly Journal of the Mythic Society*, Jan., 1926.

'*b.* 598 A.D.) was the first to enunciate* the principle of composition of roots which is tantamount to the modern principle of composition of quadratic forms.

"If $x = a$, $y = b$ be a pair of roots of $\alpha x^2 + p = y^2$, and $x = c$, $y = d$ be roots of $\alpha x^2 + q = y^2$, then $x = bc \pm ad$, $y = bd \pm \alpha ac$ is a pair of roots of $\alpha x^2 + pq = y^2$."

The following special identities are derived by Brahmagupta from the above principle.

If $x = p$, $y = q$ is a solution in integers of $Dy^2 + 4 = x^2$, then

$$x = \frac{p(p^2 - 3)}{2}, \quad y = \frac{q(p^2 - 1)}{2}$$

is a solution in integers of $Dy^2 + 1 = x^2$.

Again, if $x = p$, $y = q$ is a solution in integers of $Dy^2 - 4 = x^2$, then

$$x = (p^2 + 2)\left\{\frac{(p^2 + 3)(p^2 + 1)}{2}\right\}, \quad y = \frac{pq(p^2 + 1)(p^2 + 3)}{2}$$

is an integral solution of $Dy^2 + 1 = x^2$.

His actual words are worth quoting, especially as they show how words may be abbreviated so as to possess the power of symbols in a mathematical presentation.

चतुरतिकेऽन्त्यपदकृति स्त्र्यूनादलितान्त्यपदगुणान्त्यपद ।
मन्त्यपद कृतिर्व्येकाद्विहृताद्यपदाहताद्यपदम् ॥
चतुरूनेऽन्त्यपदकृती त्र्येकयुतेवधदलं पृथग्व्येकं ।
व्येकाद्याहतमन्त्यं पदवधगुणमाद्यमान्त्यपदम् ॥

About 1150 A.D. Bhaskara surprises us by his brilliant *Chakravala* or the cyclic method, which is outlined below.

Let the roots of the equation

$$Ny^2 + k = x^2 \qquad \dots \quad (1)$$

be $x = a$, $y = b$ (N being a positive non-square integer). Solve the linear indeterminate equation $\frac{bx + a}{k} = y$ in integers, and choose a value

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* In this remarkable principle Brahmagupta has anticipated Euler (1707—1783) by a thousand years. (Vide '*Diophantus of Alexandria*' by Sir T. L. Heath, p. 290)

l for x so that $|l^2 - N|$ may be the least possible. Then, the corresponding value for y is $\frac{bl + a}{k}$, which is the integral value for y in the new equation

$$Ny^2 + \frac{l^2 - N}{k} = x^2, \quad \ldots \quad (2)$$

the corresponding integral value for x being $(al + Nb)/k$.

Just as we derived (2) from (1), we may derive another equation from (2) with known roots and so on, until the additive ultimately reduces to ± 4, ± 2, or ± 1. In case the additives are ± 4, ± 2, or -1, the roots of the equation with unity as the additive can be obtained by the principle of composition.

The true nature and importance of the *Chakravala* has not been realised even by the eminent orientalist, Colebrooke. Relying on him, probably, H. J. S. Smith misses the essential point in the cyclic method and remarks that Bhaskara and Brahmagupta misunderstood the nature of the problem.*

I have shown† that Bhaskara's method is more in line with Gauss' solution based on the theory of quadratic forms than Lagrange's, that it leads to a new set of reduced forms which I call Bhaskara forms,‡ and has the advantage of reducing the number of recurring elements and thus shortening computation

Remarkably enough, one of Bhaskara's worked examples, *viz.*, $61y^2 + 1 = x^2$, happens to be one of those proposed by Fermat in his letter to Frenicle in February 1657.

Indeed the cyclic method deserves Hankel's unstinted praise that it is certainly the finest thing achieved in the Theory of Numbers before Lagrange. Well might Colebrooke remark: ' Had an earlier translation of Hindu mathematical treatises been made and given to the public,

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* *Collected Papers*, Vol. I, pp. 192—200.

† ' New Light on Bhaskara's *Chakravala*, etc.' by the present writer in *J. I. M. S.*, Vol. XVIII (1930).

‡ A Bhaskara form is a quadratic form (A, B, C) of positive determinant N, such that

$$A^2 + C^2/4 < N, \text{ and } C^2 + A^2/4 < N.$$

especially to the early mathematicians in Europe, the progress of mathematics would have been much more rapid, since algebraic symbolism would have reached its perfection long before the days of Descartes, Pascal, and Newton;' and, I may add, Euler and Lagrange would have had a better start in their work on continued fractions.

In deference to Brahmagupta, the so-called Pellian Equation $x^2 - Ny^2 = 1$ ought to be called the Brahmagupta equation.

In passing, I may mention that a curious little problem on the indeterminate equations

$$x + a = y^2, \quad x - b = z^2$$

persists in several Indian mathematical works and appears to have gone to Europe and thence to the New World in 1556. It occurs in the American work '*Sumario Compendioso*' of one Juan Diez* in the form

'Give me a number which increased by fifteen is a square number and decreased by four is also a square number.'

I may conclude this topic by giving a sample of an apparently complex problem in indeterminate equations solved by Bhaskara.

"Tell me quickly, mathematician, two numbers such that the cube-root of half the sum of their product and the smaller number, and the square root of the sum of their squares and those square-roots) extracted from the sum and difference increased by two, and that extracted from the difference of their squares added to eight, being all five added together may yield a square-root—excepting however 6 and 8."

Contributions to Trigonometry.

Trigonometry, in its initial stages, developed as a handmaid to Astronomy in the hands of the ancient Hindus and Greeks, and this explains the rather startling fact that Spherical Trigonometry was developed earlier than Plane Trigonometry. The familiar bow and arrow must have suggested to the speculative Indian mind their mathematical analogues. The semi-chord and the arrow gave rise to the sine and the versed-sine functions. It is interesting to note that the

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* Smith's *History of Mathematics*, Vol. I, pp. 355, 356.

semi-chord is an entirely Indian idea, for, even Claudius Ptolemaeus, the most important figure in the history of Greek Astronomy, gave formulae only for the full chords. While the Greeks divided the radius sexagesimally, the Hindus divided the circumference of a circle into 21600 minutes and thus straight away hit upon the circular measure.*

The early Hindus readily recognised the approximation $\sin\theta = \theta$ for small angles. They established the fundamental formulae corresponding to $\sin^2\alpha + \cos^2\alpha = 1$, $4\sin^2\alpha/2 = \sin^2\alpha + \text{vers}^2\alpha$, $\text{vers}\,\alpha = 2\sin^2\alpha/2$ and computed relations between the half-chords and their arcs. They managed all their astronomical formulae very ingeniously with the help of the three trigonometrical functions: the sine (ज्या)†, the versed-sine (उत्क्रमज्या) and the cosine (कोटिज्या). The fundamental formulae of spherical trigonometry corresponding to

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C};$$

$$\cos A \sin c = \cos a \sin b - \sin a \cos b \cos C;$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

were also deduced most simply from the properties of right-angled triangles and the principle of similarity.

It is remarkable that almost every important Hindu astronomical work from the *Suryasiddhanta* to the *Siddhanta Siromani* gives an interpolation (or extrapolation) formula for the tabulation of successive sines. In the *Aryabhatiyam* and the *Suryasiddhanta*, we find the extrapolation formula:‡

$$\sin(n+1)\alpha - \sin n\alpha = \sin n\alpha - \sin(n-1)\alpha - \frac{\sin n\alpha}{225} \quad \text{(approx.)}$$

where n is an integer < 24 and $\alpha = 3\frac{3}{4}^\circ$.

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* It is necessary to note that the Hindu sine is a length and not a ratio like the modern sine; it is expressed, however, in terms of the arc. The early Hindus used the arc in place of the angle, attributing to it all the properties of the angle.

† The modern 'sine' is derived from the Sanskrit 'ज्या' or 'जीव.' The Arabs adapted Sanskrit 'जीव' into their Arabic 'dschiba' which they naturalised as 'dschaib' meaning 'bosom.' Later this was translated by Plato of Trivoli into 'Sinus' whence the modern 'sine.'

‡ 'The Hindu Sine-Table' by the present writer in *J.I.M.S.*, Vol. XV, Dec, 1923.

In the Ganitadhyaya of *Siddhanta Siromani* Bhaskara gives an approximate quadratic interpolation formula which, in modern notation, runs

$$\sin(x - h\theta) = \sin x - \frac{\sin(x+h) - \sin(x-h)}{2}\theta + \frac{\sin(x+h) - 2\sin x + \sin(x-h)}{2}\theta^2 \quad (\theta \leqslant 1).$$

Brahmagupta also appears to mention the above formula in his '*Khanda Khadyaka*.'*

It is worthy of note that the terms on the right-hand side of the above approximation correspond to the first three terms in the well-known general interpolation formula due to Newton.†

Hindu Astronomy.

We shall now take a rapid glance at Hindu astronomy—the science to which all the rest of mathematics was regarded as merely auxiliary.

In the most ancient Vedic hymns, we have evidences of description of the well-known astronomical phenomena, the motions of the Sun and the Moon, the seasons, the number of days in a year, the two ayanas, Devayana and Pitriyana, the intercalary month, Rita or the zodiacal belt which is the eternal track of the luminaries and the bright planets बृहस्पति and वेन. The ancient scriptures contain references (though in legendary garb) to the zodiacal shifts ‡ of the year beginning at the vernal equinox. There are sufficient data in them for postulating the phenomenon of precession of the equinoxes and determining its rate. So it is no wonder that the Siddhantas give such accurate rates as 54″ in the *Surya-siddhanta* and 59″.9007 in Bhaskara's work where the author quotes his result from Munjala. The European critics, who have a poor opinion of the ancient Hindu observations, consider this

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* 'Brahmagupta on Interpolation' by P. C. Sengupta in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, No. 3, 1931.

† I discussed the relation betwen this formula and Newton's corresponding Indian one in a paper submitted to the 12th Indian Science Congress, Benares, 1925. Vide *Abstracts*, p. 74.

Also Ball: *Spherical Astronomy*, p. 18, where a quadratic interpolation formula is given from which Bhaskara's result may be readily deduced.

‡ Tilak's scholarly work '*Orion*' may be consulted in this connection.

a lucky hit*, for the Greeks in spite of their (supposed) systematic observations and their marvellous inductive and deductive methods were content with a very poor precessional rate of 36″ which they probably got from the Babylonians. It has now been discovered that the precession of the equinoxes was known to the Babylonian astronomer Kidinnu long before the time of Hipparchus.

From the Vedas, we pass on to the Siddhantas. At one time there appear to have been as many as twenty†, viz., *Brahma*, *Surya*, *Soma*, *Brihaspati*, *Garga*, *Narada*, *Parasara*, *Poulastya*, *Vasishta*, *Vyasa*, *Atri*, *Kasyapa*, *Marichi*, *Manu*, *Angirasa*, *Lomasha*, *Pulisha*, *Yavana*, *Bhrigu*, and *Chyavana*. These were perhaps the earliest efforts (barring *Vedangajyotisha*) at crystallising the extant astronomical knowledge in as scientific a form as possible, about 2000 years ago. Of these the *Brahma* and the *Surya* siddhantas are the most popular and have been revised by various siddhantic writers, starting from Aryabhata.

The Siddhantas mention the number of revolutions of the sun, moon, and planets and their nodes for a Mahayuga (432×10^4 years) or a Kalpa (432×10^7 years). These huge numbers entering into Hindu calculations were necessary for accurate computation even as the seven-figure tables (in other words, fractions with denominator 10^7) are used in modern calculations. The Siddhantic assumption regarding the motions of the planets is that all of them travel the same number of Yojanas in their mean circular orbits during a Kalpa, and this constant is described as *Khakaksha* (खकक्षा) the circumference of the sphere to which solar rays extend. According to Bhaskara this constant is 187, 120, 692 $\times 10^8$ while according to Suryasiddhanta it is 18, 712, 080, 864 $\times 10^6$. The inferences from this hypothesis just correspond to the findings of Newtonian astronomy, if we

* The Hindus have apparently made several such lucky hits in many other branches of mathematics, as pointed out in the course of this paper.

† *Vide* p. 274, Appendix to the Translation of the *Suryasiddhanta* by the Rev, E. Burgess (1860).

assume the celestial bodies to revolve in concentric circles under a constant acceleration directed to the centre of the earth.

Side-tracked as they were by the wrong geocentric scent, the ancient Hindus like the Greeks reconciled the apparent discrepancies in the planetary motions by the contrivance of epicycles (नीचोच्चवृत्त) and eccentrics (मन्दवृत्त and शीघ्रवृत्त) and an elaborate calculus of approximations. One remarkable point about these contrivances is that they also lead, though by circuitous route more or less to the same results as modern methods to a first degree of approximation. Possibly they may be made to yield more refined results. The Indian physical explanation of the inequalities in the planetary motion may raise a smile among the moderns, but taken figuratively it merely means that there are certain centres of perturbation in the orbit which produce variations in velocity. Indian astronomers were fairly correct in their calculations of the moon's distance from the earth but erred in the distances of the sun and the planets as the latter were calculated from the hypothesis of constant velocity.

The other points of interest in Indian astronomy are the calculation of the moments of eclipses of the sun and the moon based on correct theory, the calculation of the length of the day by means of observations of the sun-dial, heliacal risings and settings, planetary conjunctions, parallax of the sun and the moon, the phases of the moon, the co-ordinates of certain well-known stars, etc.

Just as in the modern text-books of astronomy, in every siddhantic work there is a chapter on the description of instruments. But the description generally is so meagre and often abstruse that many western critics are tempted to believe that the instruments merely served to illustrate to the student the theory underlying astronomical calculations. The fact is that the explanations are purposely made scanty. The knowledge of the instruments was regarded as a privilege to be transmitted orally by the Guru with special care to the select few of his faithful disciples. This knowledge is given the exalted name of *Jyotishopanishad.*

There is enough evidence in the texts of the great Siddhanta writers to show that they actually verified the astronomical elements by personal observation of the heavenly bodies before incorporating them in their texts. Aryabhata explains briefly how he determined the mean motions of the planets by observing the intervals between their successive conjunctions with one another as well as with any particular star. Bhaskara expressly mentions that the instruments are meant for actual observation, and shrewdly remarks that more than instruments the mind behind them is valuable : धीरेकं पारमार्थिकं यन्त्रम् ।

Of the Hindu astronomers after Bhaskara, the following deserve notice :

(1) Mahendra Suri (*b.* 1320 A.D.) was the author of *Yantraraja*, a translation of a Persian work, mainly devoted to the construction, fitting and examination of instruments. The obliquity of the ecliptic is here given to be $23^{\circ}35'$ and the rate of precession $45''$ a year. The work contains the latitudes and longitudes of 32 stars, and tables of sines and cosines for every degree.

(2) Gnanaraja (1503 A.D.) was the author of *Siddhanta Sundara.*

(3) Ganesa, a contemporary of Gnanaraja, was the author of numerous original astronomical works and commentaries. His *Grahalaghava*, supposed to be written in his thirteenth year, is a famous and popular book.

(4) Muniswara (*b.* 1603 A.D.) was a great scholar and a prolific writer on Astronomy and Astrology. His important work was *Siddhanta Sarva Bhauma.* He was an astrologer in the court of the Moghul Emperor Shah Jehan.

(5) Kamalakara (*b.* 1616 A.D.) was a contemporary and rival of Muniswara. He is well-known through the work, *Siddhanta tatva viveka*, based on the doctrines of *Surya-siddhanta.* He appears to be a good mathematical scholar versed also in Arabic works on Mathe-

matics and Astronomy. He makes frequent references to Mirza Ulugh Beg, a Persian royal astronomer of the tenth century.

(6) Jagannatha (*b.* 1652 A.D.) was a Telugu Brahmin. He was a great linguist versed in Sanskrit, Arabic, and Persian as well as a mathematician and astronomer. He was the chief pandit in the court of Jayasimha, king of Jaipur, and translated Ptolemy's ***Almagest*** from Arabic into Sanskrit. The translation entitled ***Siddhanta Samrajya*** contains also numerous citations from Ulugh Beg's work as well as from Jayasimha's original contributions. In addition it contains descriptions of various astronomical instruments set up by Jayasimha in the observatories at Benares and Ujjain. Jagannatha's ***Rekhaganita***, edited by Trivedi, is very well-known. It is the translation of Euclid's elements from Arabic into Sanskrit. Jagannatha's scholarship attracted the attention of the Emperor Aurangzeb, who appointed him chief pandit in his court.

No notable contributions appear to have been made to Hindu astronomy after the time of Bhaskara. Native astronomers had probably reached the highest level possible in the twelfth century with the amount of mathematical equipment of that age, for any further advance depended on advances in the auxiliary sciences of Mathematics, Optics, and Dynamics.

With the Muhammadan conquest of India, which began in the last quarter of the twelfth century, commenced the decay of Indian initiative in the positive sciences, and the little originality that was left after the fall of the Moghuls, was stifled by contact with Western civilization. Thus India has been passing through a gloom for the last six centuries, from which she is just recovering and attempting to make a new synthesis of her ancient civilization with the achievements of modern culture.

ON CONICS HAVING A COMMON SELF-CONJUGATE TRIANGLE.*

K. SATYANARAYANA, *Rajahmundry*.

1. Introduction.

Let S be any point in the plane of a fundamental triangle $O_1O_2O_3$, and let the perpendiculars at S to SO_1, SO_2, and SO_3 meet the opposite sides at L_1, L_2, and L_3. It is well-known that the circles on the three diagonals of a complete quadrilateral as diameters are co-axial, and from this it follows that L_1, L_2, L_3 lie on a line which may be called the ***ortho-line*** of S with respect to the triangle $O_1O_2O_3$.† It is easy to see that

for any conic having $O_1O_2O_3$ for a self-conjugate triangle, any directrix is the ortho-line of its corresponding focus ... (1.1)

On the other hand, every line has two ***ortho-points*** (*i.e.*, points of which it is the ortho-line), these being the common points of the circles on the diagonals of the quadrilateral formed by the line and the sides of $O_1O_2O_3$ as diameters. Further it may be shown that

the two ortho-points of a line are inverse points with respect to the polar circle of $O_1O_2O_3$ ... (1.2)

The object of this paper is to obtain the properties of the ortho-line-point transformation with a view to applying the results to a study of conics having a common self-polar triangle.

2. The Equation of the ortho-line.

Let O_4 be the ortho-centre and H_1, H_2, H_3 the feet of the altitudes of $O_1O_2O_3$. Taking H_1 for origin and $H_1O_2O_3$ and $H_1O_1O_4$ for co-ordinate axes, let the co-ordinates of O_1, O_2, O_3, O_4 be $(0, a_1)$, $(a_2, 0)$, $(a_3, 0)$ and $(0, a_4)$ so that $a_1a_4 + a_2a_3 = 0$. Also let the

* I am indebted to Mr. S. Purushotham, M.A., for having read this paper and made several useful suggestions.

† We shall omit this reference to the triangle $O_1O_2O_3$ in future. Unless there is any statement to the contrary, the common self-conjugate triangle is always $O_1O_2O_3$, and all ortho-lines and ortho-points have reference to this triangle.

co-ordinates of any point S be (h, k). We shall denote by P_{rs} $(r, s = 1, 2, 3, 4)$ the power of S with respect to the circle on O_rO_s as diameter, and by P_{11}, P_{22}, P_{33} the powers of S with respect to the circles having for diameter H_1O_1, H_2O_2, and H_3O_3 respectively. It will be found after an easy calculation that the equation to the ortho-line of S is

$$P_{23}\, hx + P_{14}\, ky = P_{23}P_{11} \quad \dots \quad (2.1)$$

From this equation, or directly from the definition, it follows that the singular points for the point-ortho-line transformations are the vertices and the feet of the altitudes of the fundamental triangle.

Lastly we notice that

the ortho-lines of S with respect to the four triangles formed out of an ortho-centric tetrad $O_1O_2O_3O_4$ are parallel, these being also parallel to the join of the ortho-centres of SO_rO_s and SO_tO_u $(r, s, t, u = 1, 2, 3, 4)$. ... (2.2)

For interchanging O_1 and O_4, P_{23} and P_{14} are unaltered, so that the ortho-lines with respect to $O_1O_2O_3$, and $O_4O_2O_3$ are parallel. Making other interchanges, the first part follows. Since $ST_1 = -P_{23}/k$ and $ST'_1 = -P_{14}/h$ where T_1, T'_1 are the ortho-centres of SO_2O_3, SO_1O_4 the second part follows.

3. Conics with a common self-conjugate triangle.

If S, S′ be the real foci of a conic having $O_1O_2O_3$ for a self-conjugate triangle, then S,S′ are isogonal conjugates with respect to the pedal triangle $H_1H_2H_3$. ... (3.1)

For H_1O_1, H_1O_2 are conjugate perpendicular lines through H_1. Hence they bisect the angle between the two tangents from H_1 to the conic, and hence also the angle between the focal lines H_1S, H_1S'. Further they are also equally inclined to H_1H_2 and H_1H_3. Hence H_1S, H_1S' are a pair of isogonal lines through H_1 with respect to the triangle $H_1H_2H_3$. Similar considerations apply to the other vertices H_2, H_3.

We shall now obtain the elements of the conic with a focus at S (h, k).

By (1.1) and (2.1), the directrix of the conic is

$$P_{23}hx + P_{14}ky = P_{22}P_{11} \qquad \textit{(Directrix)}$$

and hence the transverse axis is given by

$$P_{14}kx - P_{23}hy = (P_{14} - P_{23})hk \qquad (\textit{Axis})$$

The *other focus* S′ lies on this line and on $\frac{y}{x} = -\frac{k}{h}$ which is the isogonal conjugate of H_1S with respect to $H_1H_2H_3$. Hence its co-ordinates are

$$\left(\frac{P_{14} - P_{23}}{P_{23} + P_{14}}\,h,\ \frac{P_{23} - P_{14}}{P_{23} + P_{14}}\,k\right) \qquad \dots \quad (3.2)$$

If e be the *eccentricity*

$$e^2 = \frac{CS}{CX} = \frac{(P_{23}\,h)^2 + (P_{14}k)^2}{P_{23}P_{31}P_{12}} \qquad \dots \quad (3.3)$$

as may be verified.

Lastly the semi-latus rectum $l = SX \cdot e = \dfrac{2R\alpha\beta\gamma}{\sqrt{P_{12}P_{23}P_{31}}}$ ... (3.4)

where α, β, γ are the trilinear co-ordinates of S with respect to $O_1O_2O_3$.

A focus of a real conic cannot be inside one only or inside all of the circles described on the sides of a self-conjugate triangle as diameters. ... (3.5)

For from (3.4) $P_{12}P_{23}P_{31}$ should be positive if e is to be real. Hence only one of the P's or all of them cannot be negative.

4. A conic with a given self-conjugate triangle $O_1O_2O_3$ is an ellipse, parabola, or hyperbola according as $N \cdot \alpha \cdot \beta \cdot \gamma \lesseqgtr 0$, where N is the power of S with respect to the nine-point circle, and α, β, γ the trilinear co-ordinates of S with respect to the triangle $O_1O_2O_3$. ... (4.1)

From (3.3), the conic is an ellipse, parabola or hyperbola according as

$$P_{12}P_{23}P_{31} - [\,(P_{23}h)^2 + (P_{14}k)^2\,] \gtreqless 0.$$

Substituting the values of the P's and simplifying, it will be found that this is equivalent to

$$N\left(\frac{k}{a_1}\right)\left(\frac{a_1h + a_3k - a_1a_3}{a_1(a_2 - a_3)}\right)\left(\frac{a_1h + a_2k - a_1a_2}{a_1(a_3 - a_2)}\right) \gtreqless 0.$$

Since α, β, γ have the same signs as the last three factors in the above the theorem follows.

(4.1) and (3.5) enable us to map out the whole plane into regions according to the nature of the conic corresponding to the position of the focus in each region.

5. Parabolas self-conjugate with respect to $O_1O_2O_3$.

With regard to the parabolas of the system we have the following results:

The locus of the focus of a parabola with a given self-conjugate triangle is the nine-point circle; ... (5.1)

The directrix of every parabola passes through the circum-centre; ... (5.2)

The directrix is parallel to the pedal line of the focus with respect to the pedal triangle. ... (5.3)

For, from (4.1) the conic is a parabola when $N\alpha\beta\gamma = 0$. It is easily seen that when α, β, or γ vanishes, the focus lies on one of the sides, and the parabola is degenerate; so that excluding these cases, $N = 0$, that is, the locus of the focus is the nine-point circle.

The theorem (5.1) admits also of an elegant synthetic proof.* It can be shown that the medial triangle of a self-conjugate triangle of a parabola is a circumscribing triangle of the parabola. Hence the focus lies on the circum-circle of the medial triangle, that is, on the nine-point circle of the self-conjugate triangle.

Further (5.2) follows from the fact that the directrix passes through the ortho-centre of any circumscribing triangle, and the ortho-centre of the medial triangle is the circum-centre of the original triangle.

As for (5.3), since the medial triangle circumscribes the parabola, the pedal line of the focus with respect to it is parallel to the directrix. It will be found that the medial triangle and the pedal triangle are ***pedo-parallel***, ***i.e.***, the pedal lines of any point with respect to these triangles are always parallel. Hence the result (5.3).

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* I am indebted Mr. Purushotham for this proof.

The semi-latus rectum of a parabola is less than the circum-diameter of any self-conjugate triangle. ... (5.4)

It can be shown easily that the maximum value of the distance between the ortho-centre and the circum-centre of triangles inscribed in a circle of radius R is 3R, and hence the maximum value of the distance of any point on the circum-circle from the ortho-centre is 4R. Here the focus lies on the circum-circle of the medial triangle and the directrix passes through the ortho-centre. Hence the semi-latus rectum is less than SO and the theorem follows by taking the above maximum value of SO.

6. Conics of the system with a common directrix.

With a given self-conjugate triangle and a given directrix, there will be, (i) when the triangle is obtuse-angled, two conics, either both real or both imaginary; and (ii) only one real conic when the triangle is acute-angled. In case (i) the two corresponding foci are the ortho-points of the given line with respect to the triangle, and the squares of the eccentricities are as the distances of the respective foci from the ortho-centre.... (6.1)

Let S, (h, k) and S_1 (h_1, k_1) be the ortho-points of the given directrix, and let us suppose that they are real. Let d, d_1 denote O_4S, O_4S_1. It is easily seen that

$$\frac{d}{d_1} = \frac{h}{h_1} = \frac{k - a_4}{k_1 - a_4} = \frac{P_{23}}{(P_{23})_1} = \frac{P_{31}}{(P_{31})_1} = \frac{P_{12}}{(P_{12})_1} = \frac{P_{11}}{(P_{11})_1}$$

where $(P_{rs})_1$ denotes the power of S_1 with respect to the circle P_{rs}

and $$\frac{P_{14}k}{(P_{14})_1 k_1} = \frac{d^2}{d_1{}^2} \text{ and } dd_1 = a_4(a_4 - a_1)$$

$$\therefore \quad \frac{P_{12}P_{23}P_{31}}{(P_{12})_1 (P_{23})_1 (P_{31})_1} = \left(\frac{d}{d_1}\right)^3 = \frac{(dd_1)^3}{d_1{}^6} \quad \ldots \quad (6.2)$$

and is positive or negative according as dd_1 is positive or negative, *i.e.*, according as the triangle is obtuse-angled or acute-angled. Hence (i) and (ii) follow from the signs of $P_{12}P_{23}P_{31}$ and $(P_{12})_1(P_{23})_1(P_{31})_1$ by (3.5).

If e, e_1 be the eccentricities of the two conics corresponding to S, and S_1 as foci

$$\frac{e^2}{e_1{}^2} = \left(\frac{(P_{23}h)^2 + (P_{14}k)^2}{P_{12}P_{23}P_{31}}\right) \div \frac{\{(P_{23})_1 h_1\}^2 + \{(P_{14})_1 k_1\}^2}{(P_{23})_1 (P_{31})_1 (P_{12})_1}$$

But

$$\left(\frac{d^2}{d_1^2}\right) = \left(\frac{P_{23}h}{(P_{23})_1 h_1}\right)^2 = \left(\frac{P_{14}k}{(P_{14})_1 k_1}\right)^2 = \frac{(P_{23}h)^2 + (P_{14}k)^2}{\{(P_{23})_1 h_1\}^2 + \{(P_{14})_1 k_1\}^2}$$

Hence, using (6.2), we have

$$\frac{e^2}{e_1^2} = \frac{d}{d_1}.$$

When a focus is on the circum-circle of a self-conjugate triangle the eccentricity e must satisfy the inequalities :

$$2\,\frac{SO + R}{SO - R} \geqq e^2 \geqq 2\,\frac{SO - R}{SO + R}$$

S, O being the circum-and ortho-centres and R the circum radius. For any value of e between the limiting values, there are two conics, their foci being symmetrical about SO... (6.2)

To prove this, it has to be noted that the co-ortho-point of a point on the circum-circle is on the nine-point circle, and the eccentricity of the conic with a focus on the nine-point circle is unity. Application of (6.1) leads to the result

The square of the semi-latus rectum of a conic with a focus on the circum-circle of a self-conjugate triangle varies as the product of the perpendiculars from that focus on the sides of the triangle. ... (6.3)

For a point on the circum-circle, whose equation is

$$x^2 + y^2 - (a_2 + a_3)\,x - (a_1 - a_4)\,y + a_2 a_3 = 0,$$

$P_{23}.P_{31}P_{12}$ reduce to $k\,(a_1 - a_4)$,

$$(ha_2 - ka_4 - a_2 a_3) = \frac{a_2}{a_1}(ha_1 + ka_3 - a_1 a_3),$$

and

$$(ha_3 - ka_4 - a_2 a_3) = \frac{a_3}{a_1}(ha_1 + ka_2 - a_1 a_2)$$

$$P_{23}P_{31}P_{12} = (a_1 - a_4)\,\frac{a_2 a_3}{a_1^2}\,\sqrt{a_1^2 + a_2^2}\,.\,\sqrt{a_1^2 + a_3^2}\,.\,\alpha\beta\gamma$$

$$= -\,a_4\,(a_1 - a_4)\,2R\,.\,\alpha\beta\gamma.$$

Hence from (3.4), $$l^2 = \frac{4R^2\,\alpha^2\beta^2\gamma^2}{P_{12}P_{23}P_{31}} = -\frac{4R^2\,.\,\alpha\beta\gamma}{a_4\,(a_1 - a_4)}$$

which varies as $\alpha\beta\gamma$.

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

On the nature of the roots of the equations :—

$$F(x) \equiv \frac{1}{n!} + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} + \dots \dots + \frac{x^r}{(n+r)!} = 0^*.$$

and $$\Phi(x) \equiv \frac{1}{n!} + \frac{2x}{(n+1)!} + \frac{3x^2}{(n+2)!} + \dots + \frac{(r+1)x^r}{(n+r)!} = 0.$$

1. LEMMA I. *If α and β be consecutive simple roots of $f(x) = 0$, and $f'(x)$ be continuous in $\alpha < x < \beta$, then $f'(\alpha)$ and $f'(\beta)$ must be of opposite signs.*

For, if h be small and positive, $f(\alpha + h)$ and $f(\beta - h)$ must be of the same sign.

Now $$f(\alpha + h) = f(\alpha) + hf'(\alpha + \theta h), \qquad 0 < \theta < 1$$
$$f(\beta - h) = f(\beta) - hf'(\beta - \theta' h), \qquad 0 < \theta' < 1$$

Hence $f'(x)$ has opposite signs, for $x = \alpha + \theta h$ and $x = \beta - \theta' h$, however small h may be, whence from continuity, $f'(\alpha)$ and $f'(\beta)$ must have opposite signs since neither of them can vanish.

LEMMA II. *The equation $f(x) \equiv 1 + x + x^2/2! + \dots + x^r/r! = 0$ has not more than one real root.*

This is a well-known result.† We have

$$f(x) = f'(x) + \frac{x^r}{r!}.$$

Hence the result follows by Lemma I, using the fact that any real root of $f(x) = 0$ must be negative. One real root exists when r is odd, and none when r is even.

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* Mr. B. B. Bagi's Question 1525 requires a proof that $F(x) = 0$ has either one real root or none.

† Hardy : *Course of Pure Mathematics*, 2nd edition, p. 392.

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We notice also that

when r is odd, $f(x)$ is always monotonic increasing, since $f'(x)$ is always positive. Similarly when r is even, $f(x)$ is monotonic decreasing in the interval $(-\infty, \alpha)$ and increasing in $(\alpha, +\infty)$ where α is the root of $f'(x) = 0$.

LEMMA III. *If* $\dfrac{d^m}{dx^m}\{\phi(x)\} = 0$ *has k imaginary roots,* $\phi(x) = 0$ *has at least k imaginary roots.*

For proof *vide* Burnside and Panton, *Theory of Equations*, Vol. I, p. 188.

We can now prove Mr. Bagi's problem (A).

Let $F(x) = \dfrac{1}{n!} + \dfrac{x}{(n+1)!} + \dfrac{x^2}{(n+2)!} + \dots + \dfrac{x^r}{(n+r)!}$.

Then $$\frac{d^n}{dx^n}\{x^n F(x)\} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} = f(x).$$

The latter equated to zero has r or $r-1$ imaginary roots according as r is even or odd. Therefore, $x^n F(x) = 0$ and hence also $F(x) = 0$ has at least r or $r-1$ imaginary roots according as r is even or odd. The degree of $F(x)$ is r. Hence the result.

2. It is interesting to note that the function $F(x)$ is *not monotonic* in the same way as the function $f(x)$ of Lemma II. To prove this we require

LEMMA IV. *The equation*

$$g(x) \equiv 1 + x + \frac{x^2}{2!} + \dots\dots + \frac{x^{r-1}}{(r-1)!} - \frac{Ax^r}{r!} = 0,$$

where $A > 0$, *does not have more than one negative root.*

We have

$$g'(x) = 1 + x + \frac{x^2}{2!} + \dots\dots + \frac{x^{r-2}}{(r-2)!} - \frac{Ax^{r-1}}{(r-1)!}.$$

$$\therefore \qquad g(x) = g'(x) + \frac{x^{r-1}}{(r-1)!}\left(1 + A - \frac{Ax}{r}\right).$$

The result now follows by Lemma I. We shall now prove that *the equation*

$$\Phi(x) = \frac{1}{n!} + \frac{2x}{(n+1)!} + \frac{3x^2}{(n+2)!} + \dots\dots + \frac{(r+1)x^r}{(n+r)!} = 0$$

has only one real root, when r is odd,

We have

$$\frac{d^n}{dx^n}\{x^n \Phi(x)\} = 1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \dots + \frac{(r+1)x^r}{r!} = \phi(x) \text{ (say).}$$

Hence, it is sufficient if we show that the equation $\phi(x) = 0$ has only one real root when r is odd.

Now

$$\phi(x) = \frac{d}{dx}\{xf(x)\} = f(x) + xf'(x).$$

Also
$$\phi(x) = \frac{d}{dx}\{xf'(x)\} + \frac{(r+1)x^r}{r!}.$$

Hence,
$$\phi(x) = \phi'(x) - f'(x) + \frac{(r+1)x^r}{r!}.$$

If possible, suppose that $\phi(x) = 0$ has two consecutive real roots α and β, which must of course be negative.

We have

$$\phi'(\alpha) = f'(\alpha) - \frac{(r+1)\alpha^r}{r!};$$

$$\phi'(\beta) = f'(\beta) - \frac{(r+1)\beta^r}{r!}.$$

By Lemma IV, $f'(x) - \frac{r+1}{r!}x^r = 0$ has not more than one negative root.

The function $f'(r) - \frac{(r+1)x^r}{r!}$ is equal to 1 when $x = 0$, and tends to $-\infty$ when $x \to +\infty$. When equated to zero, it has therefore an odd number of positive roots always. Hence if r is odd, it has no negative root, and if r is even, it has a single negative root.

When r is odd, therefore, $\phi'(\alpha)$ and $\phi'(\beta)$ are of the same sign. Lemma I now completes the proof.

3. We can further show that *the single real root of* $\phi(x) = 0$ *when r is odd, lies between* 0 *and* -1.

For if $a_n = 1 - 2 + \frac{3}{2!} - \dots (-)^n \frac{n+1}{n!}$, it is easy to see that

$a_2, a_4, a_6, \ldots$ form a decreasing sequence, and $a_1, a_3, a_5 \ldots$ an increasing sequence, both sequences tending to the same limit which is the value when $x = -1$ of $\underset{r \to \infty}{\text{Lt}} \{ f(x) + xf'(x) \}$ that is of $e^x(1+x)$. Since the value zero, $a_1, a_3, a_5, \ldots$ are all negative. But $\phi(-1) = a_r$ which is negative when r is odd. Also $\phi(0)$ is positive. Hence the result.

4. What is the nature of the roots of the equation, $\phi(x) = 0$, *when r is even?*

The roots are not necessarily all imaginary. To prove this, we have the fact that the infinite series $1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} \ldots \ldots \ldots$ converges to $e^x(1+x)$, which is negative when $x < -1$.

When $x = x_0 < -1$, the sum of the infinite series will be negative. We can choose an even number r so large that $1 + 2x_0 + \ldots + \frac{(r+1)x_0^r}{r!}$ is negative. Hence the equation $\phi(x) = 0$, for such a choice of r, has at least one root $< x_0$, and at least one root $> x_0$.

We can, however, prove that *when r is even,* $\phi(x) = 0$ *has no root in the interval* $x \geqslant -1$,

Consider the sequence

$$b_n = 1 + 2x + \frac{3x^2}{2!} + \ldots + \frac{(n+1)x^n}{n!}$$

for values of x in the interval $(-1 \leqslant x < 0)$.

It may be shown that $b_2, b_4, b_6, \ldots$ form a decreasing sequence whose limit $e^x(1+x)$ is not negative in $(-1 \leqslant x < 0)$. b_{2n} is therefore always positive for $x \geqslant -1$.

It may be observed that the values $r = 2$ and $r = 4$ provide exceptions. The equations

$$1 + 2x + \frac{3x^2}{2!} = 0;$$

$$1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \frac{5x^4}{4!} = 0,$$

have all their roots imaginary, as may be verified by well-known tests.

If $r = 6$, the equation

$$1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots + \frac{7x^6}{6!} = 0$$

has a root in $(-1\frac{1}{2}, -1)$ and a root $< -1\frac{1}{2}$.

Similarly, when $r = 8$, there is at least one root < -2.

5. We can now answer the question whether the function in Mr. Bagi's problem (A), *viz.*

$$F(x) = \frac{1}{n!} + \frac{x}{(n+1)!} + \ldots\ldots + \frac{x^r}{(n+r)!}$$

is monotonic. When r is even, $F'(x) = 0$ has a, single real root α, say, which is negative. $F'(x)$ is negative in the interval $(-\infty, \alpha)$ and positive $(\alpha, +\infty)$. $F(x)$ is therefore monotonic decreasing and increasing respectively in these intervals.

If r is odd, but equal to 1 or 3 or 5, $\phi(x)$ is always monotonic increasing. For other odd values of r, the graph of the function $\phi(x)$ is complicated. There is at least one (and hence there are two) real points of inflexion.

6. Other results :—

(1) The equation

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + Ax^{r+2} + Bx^{r+4} + Cx^{r+6} + \ldots + Lx^{r+2k} = 0$$

where A, B, ... L are positive, has not more than one real root.

(2) The equation

$$A + x + \frac{x^2}{2!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} = 0$$

has one and only one real root for all values of A.

(3) The equation

$$-A + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} + \ldots + \frac{x^{2r}}{(n+2r)!} = 0$$

has two and only two real roots if $A > 0$. (Use § 6).

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the interest of this section by sending suitable news items of interest.

At the invitation of the Bombay University, the Eighth Conference of the Society was held on the 21st, 22nd and 23rd December, 1932, under the patronage of His Excellency the Rt. Hon'ble Sir Frederick Sykes, Governor of Bombay and Chancellor of the Bombay University. The Presidential Address was delivered by Rao Bahadur P. V. Seshu Aiyar Avl., President of the Society. Thirty-five papers were presented to the Conference, besides which there were three public lectures by Dr. M. N. Saha on "The Present Crisis in the Science of Dynamics," Dr. Vaidyanathaswamy on 'The Nature of the Continuum" and by Rao Bahadur P. V. Seshu Aiyar on "The Nature of Mathematics and Religion." There was also a business meeting, and discussions on the teaching of mathematics in the school and collegiate stages.

The Silver Jubilee Celebrations of the Society took place on the 24th December, under the presidency of the Rev. Dr. J. Mackenzie, Vice-Chancellor of the Bombay University, when a public Address was presented to M.R.Ry. M. T. Naraniengar, Avl., in appreciation of his services to the Society. A detailed report of the Conference proceedings, as also the texts of the papers presented to it, will appear in the Jubilee Memorial Volume, which is under preparation.

The joint-librarian draws attention to a phenomenon, which we believe is neither new nor rare, namely, that journals in circulation encounter unexpected resistances in their course. As these periodicals have to describe large and complicated orbits, he particularly requests the co-operation of the members in minimising these perturbations. If a periodical arrives at a time when a member is too busy to study it, the journal may be forwarded to the next in the list and may be had again from the librarian after the circulation is over.

The Ramanujam Memorial Prize of the Madras University, of the value of Rs. 500, will be awarded for the best essay or thesis written on any branch of Mathematics, embodying the result of the personal investigations of the author and containing clear evidence of independent and original research. The prize is open to all persons born or domiciled in India. Competitors should forward the manuscripts so as to reach the Registrar not later than the 1st December, 1933.

The De Morgan Medal of the London Mathematical Society has been awarded to Bertrand Russel for his mathematical work.

The gold medal of the Royal Astronomical Society has been awarded to Dr. V. M. Slipher, Director of the Lowell observatory, for his spectroscopic researches on the planets, stars and nebulae.

The title of Emeritus Professor of Mathematics in the University of London has been conferred on Professor S. A. F. White, on his retirement from the chair of mathematics at King's College, London.

Dr. Harold Spencer Jones, M.A., Sc.D., F.R.S., has been appointed Astronomer Royal in succession to Sir Frank Dyson who retired in February, 1933.

Two volumes of the *Verhandlung des internationalen Mathematiker Kongresses*, Zurich, 1932, have been issued. The first volume contains a report of the congress proceedings and also the public addresses by Profs. Fueter, Caratheodory, Cartan, Bieberbach, Morse, Noether, Bohr, Severi, Nevanlinna, Wavre, Alexander, Riesz, Valiron, Sierpinski, Bernstein, Menger and Stenzel. The second volume contains abstracts of sectional papers.

Prof. E. W. Hobson, emeritus professor (formerly Sadlerian Professor) of Pure Mathematics at Cambridge, died on the 18th of April, aged 76. His name will be familiar to the younger students as the author of a standard treatise on Plane Trigonometry and to those of maturer years by his classical Theory of Functions of a Real Variable.

Prof. E. H. Moore, Emeritus Professor of the University of Chicago, and for sometime President of the American Mathematical Soeiety, died on 30th December, 1932, aged 70 years.

BOOKS RECEIVED FOR REVIEW.

Intermediate Trigonometry, by V. RANGANATHAN, B.A. (Hons.), and G. V. RAMACHANDRAN, B.A. (Hons.), St. Joseph's College, Trichinopoly, with a Foreword by Prof. R. SRINIVASAN, M.A., Trivandrum, pp. 225, Wednesday Review Press, Trichinopoly. Price Re. 1-12-0.

Functions of a Complex Variable, by THOMAS MACROBERT, M.A., D.Sc., Professor of Mathematics, Glasgow University, Macmillan, 1933, Second Edition, pp. 347. Price 14/-.

GLEANINGS.

1. As the prerogative of Natural Science is to cultivate a taste for observation so that of Mathematics is, almost from the starting point, to stimulate the faculty of invention. —*Sylvester.*

TRNSFINITE ARITHMETIC.

2. Om, whole is that, whole (too) is this; from whole, whole cometh; take whole from whole, whole remains. —*Isa Upanishad.*

THE UNIVERSAL LAW OF ATTRACTION.

3. The attendance at a Professor's lectures varies directly as the Professor's power of exposition and inversely as the square of the number of lectures delivered ...... —*A Text-book of Intermediate Algebra.*

SOLUTIONS TO QUESTIONS.

The sections on "Solutions" and "Questions" are continued from the JOURNAL OF THE INDIAN MATHEMATICAL SOCIETY, *Vol. XIX. Preference will be given to such solutions as are not mere verifications, but seek to examine critically, generalise, or correlate the results with those in other fields.*

Question 1553.

(C. N. SRINIVASIENGAR):—Prove the following properties of Steiner's quartic surface:

(1) The parabolic curve consists of three parabolas, and constitutes the envelope of the asymptotic lines;

(2) There exists a quartic curve on the surface the indicatrix at any point of which is a plane cubic.

Solution by the Proposer.

Steiner's quartic surface may be represented parametrically in the form*

$$x = A(p-a)^2(q-a)^2,$$
$$y = B(p-b)^2(q-b)^2,$$
$$z = C(p-c)^3(q-c)^2,$$

where A, B, C, a, b, c are constants.

The values of the fundamental magnitudes L, M, N of the second order are given by

$$L = \frac{-4ABC}{V}(b-c)(c-a)(a-b)(p-q)^2(q-a)(q-b)(q-c),$$
$$M = 0,$$
$$N = \frac{-4ABC}{V}(b-c)(c-a)(a-b)(p-q)^2(p-a)(p-b)(p-c),$$

where $V^2 = EG - F^2$.

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* *Vide* Forsyth: *Differential Geometry*, § 50.

The parabolic curve is included in L = 0, and N = 0. Equating L and N to zero, we get

(i) $p = q$.

(ii) $q = a$. (v) $p = a$.

(iii) $q = b$. (vi) $p = b$.

(iv) $q = c$. (vii) $p = c$.

The factor $p - q$ makes L, M, N all to vanish. It gives the quartic curve

$$x = A(p-a)^4$$
$$y = B(p-b)^4$$
$$z = C(p-c)^4.$$

Not being a double line, this must represent the locus of points at which the indicatrices are plane curves of degree higher than two.

(ii) and (v) give the same curve; similarly (iii) and (vi); and (iv) and (vii).

(ii) gives the section of the surface by the plane $x = 0$. The equations to this section may be obtained by using the cartesian equation of the surface, *viz.*,

$$\left(\frac{x}{A}\right)^{\frac{1}{2}}(b-c) + \left(\frac{y}{B}\right)^{\frac{1}{2}}(c-a) + \left(\frac{z}{C}\right)^{\frac{1}{2}}(a-b) = -(b-c)(c-a)(a-b)$$

or otherwise, in the form

$$x = 0; \left(\frac{y}{B}\right)^{\frac{1}{2}}(c-a) + \left(\frac{z}{C}\right)^{\frac{1}{2}}(a-b) = -(b-c)(c-a)(a-b).$$

The section is a parabola.

The equation of the asymptotic lines, *viz.*

$$L\,dp^2 + 2\,M\,dpdq + N\,dq^2 = 0$$

reduces to

$$(q-a)(q-b)(q-c)\,dp^2 + (p-a)(p-b)(p-c)\,dq^2 = 0.$$

It is evident that the curve $p = a$ satisfies this equation. The three parabolas constituting the parabolic curve are thus asymptotic lines, *i.e.*, any tangent to one of these parabolas is an inflexional tangent to the surface at the point of contact, and hence touches the skew asymptotic line passing through the point.

Question 1560.

(B. RAMAMURTHI, M.A.):—The normals at A, B, C to a circum-conic are concurrent at O. $A_1B_1C_1$ is the pedal triangle of O with respect to ABC. $A_2B_2C_2$ is the pedal triangle of O with respect to $A_1B_1C_1$ and so on. Show that there is a conic touching the sides of $A_nB_nC_n$ at A_{n+1}, B_{n+1}, C_{n+1} the normals at which are concurrent at O.

Solution by A. Ranganatha Rao and K. V. Vedantham.

It is obviously sufficient to prove that there exists a conic touching the sides of ABC at $A_1B_1C_1$, for the same arguments repeated for the successive triangles in question, lead to the desired result. In other words, we are required to show that AA_1, BB_1, CC_1 are concurrent.

Now, let the tangents at ABC to the circum-conic for which AO, BO, CO are normals form a triangle PQR, so that we have PA, QB, RC concurrent. From the similar triangles OA_1C, OBP we have

$$\frac{OA_1}{A_1C} = \frac{OB}{BP},$$

and from the similar triangles OBA_1, OPC, we similarly have

$$\frac{BA_1}{OA_1} = \frac{PC}{OC}.$$

$\therefore$ multiplying $$\frac{BA_1}{A_1C} = \frac{OB \cdot PC}{BP \cdot OC}.$$

Writing down the two other similar equalities and multiplying we have

$$\frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_1}{C_1B} = \frac{OB \cdot PC}{BP \cdot OC} \cdot \frac{OC \cdot QA}{CQ \cdot OA} \cdot \frac{OA \cdot RB}{AR \cdot OB}$$

$$= \frac{PC \cdot QA \cdot RB}{BP \cdot CQ \cdot AR} = 1.$$

$\therefore$ AA_1, BB_1, CC_1 are concurrent.

Also solved analytically by B. Ramamurthy and V. A. Mahalingam.

Question 1564.

(T. S. RAMANUJAN, B.A. (Hons.):—If p, q, r are the lengths of the tangents drawn from the vertices A, B, C respectively of any triangle inscribed in a circle centre O, to another circle centre O′, prove that

$$19\ \Delta^2 l^2 = \Sigma\, a^2 (p^2 - q^2)(p^2 - r^2)$$

where Δ is the area of the triangle ABC, l the distance OO′, and a, b, c are the sides of the triangle ABC.

Solution by S. Narayanan.

Taking ABC as the triangle of reference, the equation of the circle centre O′ in areal co-ordinates is

$$(p^2x + q^2y + r^2z)(x + y + z) - (a^2yz + b^2zx + c^2xy) = 0,$$

and the radical axis of this circle and the circle ABC is

$$p^2x + q^2y + r^2z = 0.$$

∴ p^2, the square of the tangent from A on circle ABC to the circle with centre O′ is 2OO′ × the perpendicular from A upon the radical axis, *i.e.*

$$2\mathrm{OO}' . \frac{2p^2\ \Delta}{\sqrt{\Omega}}$$

where

$$\Omega = p^4a^2 + q^4b^2 + r^4c^2 - 2q^2r^2bc \cos A - 2r^2p^2 \cos \cos B - 2p^2q^2ab \cos c.$$

Hence $\qquad \Omega = 16\ \Delta^2 . \mathrm{OO}'^2$

that is

$$\begin{aligned} 16\ \Delta^2 . l^2 &= \Sigma\, a^2p^4 + \Sigma\, q^2r^2 (a^2 - b^2 - c^2) \\ &= \Sigma\, a^2 (p^4 + q^2r^2 - p^2r^2 - p^2q^2) \\ &= \Sigma\, a^2 (p^2 - q^2)(p^2 - r^2) \end{aligned}$$

Also solved by T. Manivannan, S. P. Ranganadachar, Y. G. Krishnaswamy, A. L. Shaikh, M. L. Chandratreya, K. M. Varghese, K. V. Krishna Sastri, N. K. Narasimhamurthi, and V. A. Mahalingam.

Question 1567.

(A. A. KRISHNASWAMY AYYANGAR, M.A., L.T.) :—If $[x]$ denotes the greatest integer in x, and n is any positive integer, show that

$$\text{(i)} \quad \left[\frac{n}{6}\right] + \left[\frac{n}{24} + \frac{1}{3}\right] + \left[\frac{n}{24} + \frac{2}{3}\right]$$

$$= \left[\frac{n}{24} + \frac{1}{4}\right] + \left[\frac{n}{24} + \frac{1}{2}\right] + \left[\frac{n}{24} + \frac{3}{4}\right] + \left[\frac{n}{8}\right]$$

$$\text{(ii)} \quad \left[\frac{1}{2} + \sqrt[4]{n + \frac{1}{16}}\right] = \left[\frac{1}{2} + \sqrt[4]{n + \frac{1}{2}}\right]$$

$$\text{(iii)} \quad \left[\frac{1}{2} + \sqrt[5]{n + \frac{13}{32}}\right] = \left[\frac{1}{2} + \sqrt[5]{n + \frac{7}{16}}\right].$$

Remarks by T. Totadri Iyengar.

On pages 281 and 285 of Vol. XVIII of *J. I. M. S.*, first part Mr. A. A. Krishnaswamy Ayyangar has proved that

$$\left[\frac{n}{p}\right] = \left[\frac{n}{pq}\right] + \left[\frac{n+p}{pq}\right] + \ldots\ldots + \left[\frac{n+p(q-1)}{pq}\right] \quad \ldots \quad (1)$$

p, q being any two positive integers, and

$$\left[\frac{1}{2} + \left(n + \frac{q}{2^p}\right)^{1/p}\right] = \left[\frac{1}{2} + \left(n + \frac{q+1}{2^p}\right)^{1/p}\right] \quad \ldots \quad (2)$$

p being any integer, and q an odd integer less than 2^p.

If in (1), we put for (p, q), successively (6, 4) and (8, 3), we obtain the result (i) on eliminating $\left[\frac{n}{24}\right]$.

If in (2), we put $p = 4$, $q = 7$, we get the result (ii) in the correct form, namely

$$\left[\frac{1}{2} + \sqrt[4]{n + \frac{7}{16}}\right] = \left[\frac{1}{2} + \sqrt[4]{n + \frac{1}{2}}\right].$$

(As given, there is clearly a mistake in the result).

If in (2), we put $p = 5$, $q = 13$, we obtain the result (iii).

Question 1569.

(R. VAIDYANATHASWAMY) :—(1) Meaning by a ***pedal tetrad*** on a central conic, a set of four points the normals at which are concurrent, show that if $a\,bcd$, $a\,b'c'd'$ are pedal tetrads, then the triads bcd, $b'c'd'$ are apolar.

(2) If the normals at b, c to a central conic pass through the centre of curvature at a, show that bc is the first polar of a in regard to the three other vertices of the inscribed rectangle which has one vertex at a.

(3) Show that the four other points, the centres of curvature at which lie on the normal at a, form a self-apolar pedal tetrad.

Solution by B. Ramamurti.

(1) From the relations $\Sigma\, t_1 t_2 = 0$, and $t_1 t_2 t_3 t_4 + 1 = 0$ connecting the parameters of a pedal tetrad, it is clear that pedal tetrads form a net apolar to the inscribed rectangles. The latter form a syzygetic pencil of tetrads, which are given by

$$t^4 + 6\lambda t^2 + 1 = 0 \qquad \ldots \quad (1)$$

so that if α be a vertex of a rectangle, the others are $-\alpha$, $1/\alpha$ and $1/\alpha$.

If $$(t - \alpha)\,(a_0 t^3) + 3a_1 t^2 + 3a_2 t + a_3) = 0$$

be apolar to (1) for all values of λ, then

$$\frac{a_0}{a_3} = \frac{a_2}{a_1} = \alpha. \qquad \ldots \quad (2)$$

Similarly if $$(t - \alpha)\,(b_0 t^3 + 3b_1 t^2 + 3b_2 t + b_3) = 0$$

be also apolar to (1) $$\frac{b_0}{b_3} = \frac{b_2}{b_1} = \alpha \qquad \ldots \quad (3)$$

Hence the apolar invariant of the two cubics a_x^3, b_x^3 namely,

$$a_0 b_3 - 3a_1 b_2 + 3a_2 b_1 - a_3 b_0$$

vanishes in virtue of the relations (2) and (3).

(2) Let the pedal tetrad which contains α twice (*i.e.*, the feet of normals from the centre of curvature at α) be

$$(t-\alpha)^2\,(c_0t^2+2c_1t+c_2).$$

Since this is to be apolar to (1)

$$b_0+b_2\alpha^2=0;$$
$$b_2+b_0\alpha^2-4\alpha b_1=0,$$

so that the residual quadratic is

$$t^2+\frac{\alpha^2-1/\alpha^2}{2\alpha}\,t-\frac{1}{\alpha^2}=0,$$

which is easily seen to be the first polar of α with respect to the cubic

$$\left(t+\alpha\right)\left(t+\frac{1}{\alpha}\right)\left(t-\frac{1}{\alpha}\right)=0.$$

(3) If the centre of curvature at β lies on the normal at α, α should be a root of the residual quadratic of β, namely, of

$$\alpha^2+\frac{\beta^2-1/\beta^2}{2\beta}\,\alpha-\frac{1}{\beta^2}=0.$$

Hence

$$\beta^4\alpha+2\beta^3\alpha^2-2\beta-\alpha=0.$$

It is easily seen that this quartic in β is both self-apolar and apolar to (1) for all values of λ and is hence a pedal tetrad.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1638. (P. V. SUBBA RAO) :—Let a and b be two integers prime to each other, and N any integer. Let q_1 and r_1 be the quotient and the remainder when Na is divided by b, q_2, and r_2, the quotient and remainder, when $q_1 a$ is divided by b, and so on, the quotient at each stage being multiplied by a and divided by b to get the next quotient and remainder. If this be done k times, show that

(i) if $N \leqslant b^k$, the sequence of remainders, $(r_1, r_2, \ldots\ldots r_k)$ are different for different numbers ;

(ii) if $A_1, A_2 \ldots \ldots A_k$ are values of N for which the sequence of remainders are (1, 0, 0, 0...), (0, 1, 0, 0...),...(0, 0, ... 1) respectively, then the number $N \leqslant b^k$ for which the remainders are $(r_1, r_2, \ldots r_k)$ is

$$r_1 A_1 + r_2 A_2 \ldots \ldots + r_k A_k.$$

(iii) Show how to determine the numbers $A_1, A_2 \ldots A_k$ in any particular case, say for $a = 4$, $b = 3$, $k = 4$.

***1639.** (K. SATYANARAYANA) :—Show that the locus of the centre of a conic with axes in given directions, and for which a given triangle is self-conjugate, is the rectangular hyperbola passing through the vertices whose asymptotes are parallel to the axes of the conics.

1640. (C. N. SRINIVASIENGAR and M. BHIMASENA RAO) :—

(i) Let $f(x, y, z) = 0$ be the equation of any conic in areal co-ordinates. If $\mathbf{F}(x, y, z) = 0$ be the areal eqution of any other curve, interpret the equation $F(X, Y, Z) = 0$, where $X = \dfrac{\partial f}{\partial y} - \dfrac{\partial f}{\partial z}$, etc.

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* 1639 in *J.I.M.S.*, Dec. 1932, is a misprint for 1637.

(ii) Hence deduce that the eccentricity e of the general conic is given by

$$\frac{(2-e^2)^3}{1-e^2} = \frac{(aa_0^2 + bb_0^2 + cc_0^2 - 2fb_0c_0 \cos A_0 - 2gc_0a_0 \cos B_0 - 2ha_0b_0 \cos C_0)^2}{4\Delta^2(A + B + C + 2F + 2G + 2H)}$$

where Δ is the area of the triangle of reference, and a_0, b_0, c_0 are its sides.

(ii) If ϕ be the angle between the equi-conjugate diameters of the Steiner in- or circum-ellipse of a triangle whose Brocard angle is ω, prove that

$$\sin\phi = \tan 60^\circ \tan\omega.$$

(iv) Deduce from (i), the following generalisation of a result of Prof. M. T. Naraniengar, *Jour. of Ind. Math. Soc.*, Vol. XV, p. 51) :—

If $\phi(x, y, z) = 0$ be the equation of a pair of conjugate diameters of the conic $f(x, y, z) = 0$, $\phi(X, Y, Z) = 0$ denotes the same lines.

1641. (C. N. SRINIVASIENGAR) :—A plane curve of degree n has δ nodes, k_1 keratoid cusps, k_2 tacnodes, and k_3 ramphoid cusps. Prove that :—

(i) $m = n(n-1) - 2\delta - 3k_1 - 4k_2 - 5k_3$.

(ii) $n = m(m-1) - 2\tau - 3i - 4k_2 - 5k_3$.

(iii) $i = 3n(n-2) - 6\delta - 8k_1 - 12k_2 - 15k_3$.

(iv) $i - k_1 = 3(m - n)$.

1642. (C. N. SRINIVASIENGAR) :—At any point on a non-torsal generator of a scroll, the scroll and the limiting conicoid through the given generator and two generators consecutive to it, have the same principal radii of curvature.

Also if any two surfaces touch along a generator, their Gaussian curvatures at any point along it are equal.

1643. (A. A. KRISHNASWAMI AYYANGAR) :—O is a fixed point; P, Q are a point-pair belonging to a given involution on a line AB. Prove that the perpendiculars from P, Q on OQ, OP envelope a parabola.

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$$F(x) \equiv \frac{1}{n!} + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} \dots \dots + \frac{x^r}{(n+r)!} = 0$$

$$\Phi(x) \equiv \frac{1}{n!} + \frac{2x}{(n+1)!} + \frac{3x^2}{(n+2)!} \dots \dots + \frac{(n+1)x^r}{(n+r)!} = 0$$

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